Contact force distribution in a pile of rigid disks

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Contact force distributions are calculated for a number of small piles of rigid disks. The smooth and the rough disk cases are discussed. It is found that the distributions are (a) functions of the mode of construction of the pile and (b) extremely sensitive to small changes or imperfections in the shape of individual disks. The so called top down algorithm is shown to apply only to a very small subset of the permitted physical contact force distributions.

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I. INTRODUCTION

The results of computer simulations for the packing of rigid spheres under gravity in three dimensions and rigid disks in two are of interest in a variety of areas—liquid and glassy states and various granular systems [1–4]. Typically, the properties calculated from these simulations have been geometrical in nature—density, average number of contacts, radial distributions, and size distributions of interstices. More recently there have been some attempts to calculate the contact force distributions in two-dimensional piles of blocks and disks [5–11]. These results have potentially important practical applications in the building and confinement of piles of granulated materials. The theoretical modeling of force distributions was stimulated by the rather surprising experimental results of Smid and Novosad [12]. These authors carried out normal and shear stress measurements in conical piles of quartz sand and fertilizer. In each case they found that across any horizontal plane the maximum values of the normal and the shear stresses occurred at points well away from the vertical axis through the apex. Much of the theoretical effort has gone into an attempt to understand the physical basis for this result. The case of the equilibrium of piles of rigid, rough disks and blocks under gravity have been simulated by Bagster and co-workers, [5–8], and elastic, rough disks by Liffman, Chan, and Hughes [9].

The analysis of the equilibrium contact force distribution in a pile of rigid disks is characterized by the problem of indeterminancy—there are more unknowns (reactions), than physical conditions (disk equilibrium equations). Bagster and co-workers [5–8] have tackled this problem by adopting the so called top down algorithm. In this technique one solves the equilibrium conditions of the disks in sequence from the top down. The indeterminancy problem is avoided by restricting the discussion to structures with at most two support points per disk and by assuming limiting friction at all contact points.

In this paper the contact force distribution in small piles of rigid disks in equilibrium under gravity is discussed. We describe two cases, (a) a symmetric pile of ten smooth disks and (b) an asymmetric pile of eight rough disks. We find that the presence of the degrees of freedom associated with the indeterminancy is related to the physical fact that the contact force distribution depends on how the pyramid was built. In other words, the problem of calculating the contact forces in a pile of disks is determinate when the equilibrium conditions and the details of the building process are given. Our results also show that (i) the contact force distribution is very sensitive to small geometrical changes or imperfections in the shapes of the disks and (ii) the top down approach is applicable only in a very small subset of possible cases.

We deal with the smooth, rigid disk case in Sec. II and the rough, rigid disk case in Sec. III. A brief summary and discussion is given in Sec. IV.

II. SMOOTH DISKS

Consider a two-dimensional pyramid made up of ten similar, smooth, rigid disks held in equilibrium under gravity by one horizontal and two vertical supports, see Fig. 1. The centers of the outermost disks lie on the sides of an equilateral triangle. We label the disks a to j, the disk-disk contacts 0 to 16, and the disk-support contacts $w1$ to $w4$, L and R, Fig. 1. Since the disks are assumed to be smooth, each contact force acts inwardly along the normal at the point of contact. We denote the magnitudes of the contact forces by R_a, where a is the label of the corresponding point of contact. Each disk is assumed to be in equilibrium so that the appropriate contact and gravitational forces add to zero. For example, in the case of the disk b, the horizontal and vertical conditions for equilibrium are

$$\tfrac{1}{2}(R_2-R_3-R_0)-R_1=0 , \tag{1}$$

$$\frac{\sqrt{3}}{2}(R_2+R_3-R_0)-w=0 , \tag{2}$$

where w is the weight common to all the disks. For disk

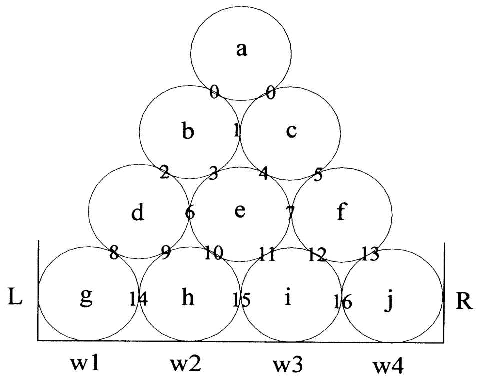

FIG. 1. The pyramid of ten, smooth, rigid disks. Each disk is labeled $a-j$. The disk-disk contacts are labeled 0–16 and the disk-support contacts labeled $w1-w4$, L and R.

g, the equilibrium conditions are

$$\tfrac{1}{2}R_8+R_{14}-R_L=0\ , \tag{3}$$

$$\frac{\sqrt{3}}{2}R_8-R_{w1}+w=0\ . \tag{4}$$

The 10 disks generate 20 equilibrium conditions. There are 24 contact forces. However, in the case of disk a, the solution to the equilibrium condition gives unique contact forces at 5 o'clock and 7 o'clock with a common magnitude $R_0=w/\sqrt{3}$. The remaining nine disks have 18 equilibrium conditions and 22 contact force magnitudes. Thus, the equilibrium state has four degrees of freedom [13] and hence the equilibrium problem posed above is insoluble as it stands. Additional assumptions or conditions are required if one is to obtain a unique solution. In what follows we shall discuss some of the kinds of assumptions we can make and the nature of the resulting equilibrium states.

Given 22 R_a and 18 linear relations, we can choose four parameters as independent variables and write the equilibrium relations as linear inhomogeneous equations for the dependent parameters in the form $A\mathbf{V}=\mathbf{W}$, where A is an 18×18 matrix, $\mathbf{V}$ is a vector whose components are the 18 dependent variables, and $\mathbf{W}$ is a vector whose components are linear combinations of the four independent variables. Not all choices of four independent variables allow us to solve for $\mathbf{V}$ in terms of $\mathbf{W}$. A case in point is the set R_1,R_2,R_3,R_L. Here A has two rows of zeroes and hence it is singular. In physical terms the inability to solve for this $\mathbf{V}$ stems from the fact that we are attempting to prescribe three of the four forces acting on b, when there are only two equilibrium conditions for this disk, see (1) and (2). In the remainder of our discussion in this section we shall treat R_3,R_9,R_{10},R_L as independent quantities. In this case A is nonsingular. Using the symbolic mathematics program MAPLE, we obtained the following solution to the 18 equilibrium conditions.

$$R_1=\frac{w}{\sqrt{3}}-R_3\ , \tag{5}$$

$$R_2=\sqrt{3}w-R_3\ , \tag{6}$$

$$R_4=R_3\ , \tag{7}$$

$$R_5=\sqrt{3}w-R_3\ , \tag{8}$$

$$R_6=\frac{w}{\sqrt{3}}-R_9\ , \tag{9}$$

$$R_7=R_{10}-R_9-R_3\ , \tag{10}$$

$$R_8=\frac{5w}{\sqrt{3}}-R_9-R_3\ , \tag{11}$$

$$R_{11}=\frac{2w}{\sqrt{3}}-R_{10}+2R_3\ , \tag{12}$$

$$R_{12}=\frac{w}{\sqrt{3}}-R_{10}+R_9+R_3\ , \tag{13}$$

$$R_{13}=\frac{4w}{\sqrt{3}}+R_{10}-R_9-2R_3\ , \tag{14}$$

$$R_{14}=R_L-\frac{5w}{\sqrt{3}}+R_9+R_3\ , \tag{15}$$

$$R_{15}=R_L-\frac{5w}{\sqrt{3}}+(2R_9+R_3-R_{10})/2\ , \tag{16}$$

$$R_{16}=R_L-\frac{2w}{\sqrt{3}}+(R_9+2R_3-R_{10})/2\ , \tag{17}$$

$$R_{w1}=\tfrac{7}{2}w-\frac{\sqrt{3}}{2}(R_9+R_3)\ , \tag{18}$$

$$R_{w2}=w+\frac{\sqrt{3}}{2}(R_9+R_{10})\ , \tag{19}$$

$$R_{w3}=\tfrac{5}{2}w+\frac{\sqrt{3}}{2}(R_9-2R_{10}+3R_3)\ , \tag{20}$$

$$R_{w4}=3w-\frac{\sqrt{3}}{2}(R_9+2R_3-R_{10})\ , \tag{21}$$

$$R_R=R_L\ . \tag{22}$$

As a check on these results, we note that the two horizontal components of the applied forces balance [Eq. (22)] and the horizontal plane exerts a vertical force $R_{w1}+R_{w2}+R_{w3}+R_{w4}=10w$ [Eqs. (18)–(21)] to support the net weight of the 10 disks.

The independent quantities R_3, R_9, R_{10}, R_L span a four-dimensional parameter space. Each point in this space describes a solution to the 18 equilibrium conditions and conversely, every equilibrium state is associated with a point in the parameter space. Because we are dealing with rigid disks, there is also the physical requirement

$$R_a\geq 0\ \text{ for all } a\ . \tag{23}$$

As a result, not all parameter space points correspond to physically realizable equilibrium states. While this condition [Eq. (23)] places a strong constraint on the number of physically permitted states, it does not reduce this number to unity, i.e., there is no unique equilibrium state for this pyramid (see below). We note in passing that the uniqueness theorems common in the physics literature apply to systems where the forces of interaction are conservative. In the case of the pyramid of rigid disks, the contact forces are not conservative and so the lack of uniqueness is to be expected.

We now show how, by building the pyramid in different ways, we can generate different contact force distributions.

(A) In the first case we start by setting the lowest four disks g,h,i,j into the container. Next the three disks d,e,f are placed on top of the bottom row. At this stage the contact forces at the two contacts labeled 6 and 7 are zero, $R_6=R_7=0$ [see Fig. 2a]. The remaining three disks a,b,c are assembled with a supported by b and c, and zero contact force between b and c, $R_1=0$. Finally, this group is placed symmetrically on the row d,e,f. This placement does not change the zero contact force conditions

$$R_1=R_6=R_7=0 \ . \qquad (24)$$

The argument for this conclusion is straightforward; as a,b,c are added to the pyramid, d,e do not exert a net horizontal force on b; e,f do not exert a net horizontal force on c; therefore, the contact force R_1 remains zero. For R_6, we note that b,c do not exert a net horizontal force on e, whereas b exerts a net horizontal force to the left on d. Thus, the effect of b on d is to cause an increase in R_8 and leave a vanishing contact force at three o'clock on d. A similar argument applies to the contact at nine o'clock on f. The 3 conditions in Eq. (24) combined with the 18 equilibrium conditions give an equilibrium state described by 1 free parameter, namely, R_L. Solving Eqs. (5)–(22) plus (24), we find the set of contact force magnitudes, which are shown in Table I under column A. For example,

$$R_5=2w/\sqrt{3}, \quad R_{15}=R_L-2w/\sqrt{3}, \quad R_{21}=5w/2 \ . \qquad (25)$$

The size R_L ($\equiv R_R$) is determined by the forces exerted by the vertical supports. This parameter affects only the forces at the horizontal, colinear contact points, 14,15,16. Because of the physical requirement $R_{15}\geq 0$, we must have $R_L \geq 2w/\sqrt{3}$, a fact indicated in the last row of Table I.

(B) In this building process we construct the pyramid symmetrically, leaving out the disk e, Fig. 2(b). This structure is in equilibrium, with disks b and c leaning inwards for support, i.e, $R_1\neq 0$. Now we put e into the pile. Since the upper disks a,b,c are already fully supported, it follows that the four contact forces R_3,R_4,R_6,R_7 vanish. Since $R_3=R_4$ [Eq. (7)], we get three new conditions, namely,

$$R_3=R_6=R_7=0 \ . \qquad (26)$$

The solution to Eqs. (5)–(22) plus (26) is shown in column B in Table I. We note that the contact force distributions are quite different in these two cases, (A) and (B); thus, the contact force distribution is a function of how the pyramid was constructed, i.e., the distribution is history dependent.

Consider yet another building process. Let us first of all shave a small portion from disk e at 11 o'clock and construct the pyramid following prescription (A), see Fig. 2(c). Then we have $R_6=R_7=0$, $R_3=0$, and $R_1\neq 0$. The latter two conditions arise because of the shaving of e. This set of vanishing contact forces coincides with case (B), see (26), and so the force distribution is described by column B in Table I. From this example, we see that in practical situations, the force distribution can be very sensitive to small imperfections in the disks, which make up the pyramid. We can think of this as geometrical chaos.

(C) Consider the pyramid with disk h missing, Fig. 2(d). For this construction

$$R_9=R_{10}=R_{14}=R_{15}=0 \ . \qquad (27)$$

However, combining (27) with (10), we conclude that $R_7=-R_3$, in contradiction with the rigid disk property [Eq. (23)]. To show why the structure in Fig. 2(d) is physically impossible, we start by constructing the semipyramid shown in Fig. 2(e). Here disks a, c, and h are missing. This structure is physically feasible; the corresponding contact forces are shown in Table I under column C: disks d and e are supported by the mutual contact at 6. In this case we are not free to vary the parameter R_L. It has the value $\sqrt{3}w/2$ determined by the conditions for equilibrium.

Now add disk c to the pyramid. Additional horizontal and vertical force components are imposed on e. Disk d cannot provide the needed compensating, horizontal in-

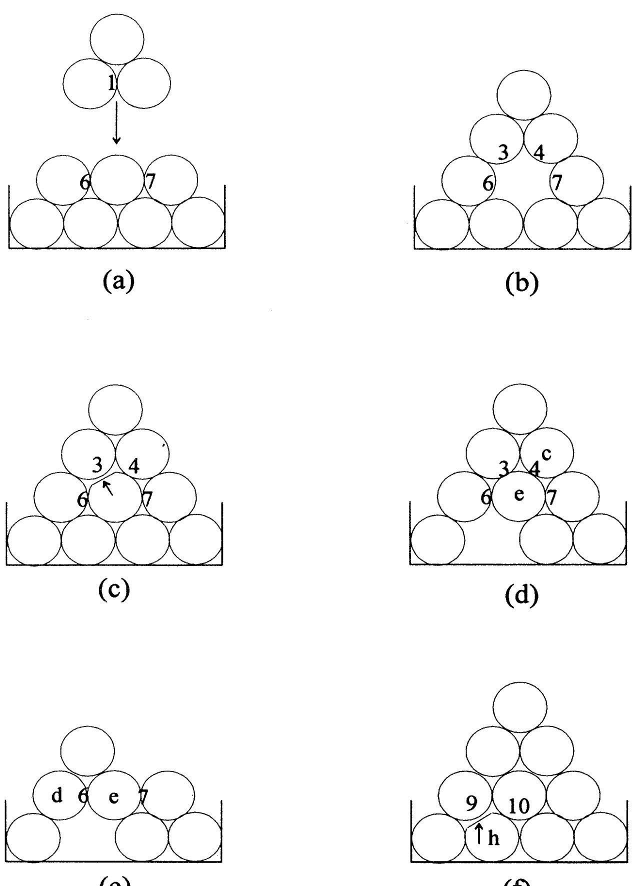

FIG. 2. Various arrangements of smooth, rigid disks (see the text and Table I.)

crease in R_6 and so moves. Thus, there is no pyramidal equilibrium. The mathematical equilibrium solution compensates by introducing a negative or attractive contact force R_7, a physically unacceptable result.

(D) Here we shave disk h at 11 o'clock and use it plus nine other perfect disks to construct a pyramid following the process described in (A) above. The presence of h allows the possibility of a contact force R_{10}. This avoids the difficulties seen in (C). The following conditions then hold:

$$R_1=R_7=R_9=0 \ . \tag{28}$$

The solution to (5)–(22) plus (28) is shown in Table I under column D. We note that the tiny geometrical flaw in h leads to (a) a significant asymmetry in the distribution of the disk-disk interactions, see in particular the variation among the supporting forces, R_{w1} to R_{w4} and (b) a significant increase in the minimum allowed value of R_L. Just as in case (B), we can replace the shaved disk h in the equilibrium pile with a perfectly round disk and leave the equilibrium force distribution unchanged. Thus, column D in Table I describes yet another force distribution in the ten disk pyramid, Fig. 1.

What we have done in the previous paragraphs is show that there are lines (A), (B), and (D) in the R_3,R_9,R_{10},R_L space, which describe physical equilibrium states of a pyramid of ten perfect disks. Clearly we can obtain additional lines by setting other combinations of normal reactions to zero. An exhaustive procedure will then produce an ensemble of equilibrium states for this pile. See further comments below.

III. ROUGH DISKS

Consider the pile in Fig. 3. Disks a and h are missing. This is the arrangement that proved to have no physical equilibrium state when the disks are smooth, Sec. II, (C). We now show that physical equilibrium states can exist if the disks are rough. There are 17 contact points in Fig. 3 and so 34 force components (17 normal and 17 frictional) to be determined. There are 8 discs with 24 equilibrium conditions (2 translational and 1 rotational per disk). Hence, we have 10 degree of freedom. To reduce some of the algebraic complexity, we shall assume that the disk-support interaction is smooth. The number of degrees of freedom then drops to five. The frictional force magnitudes are denoted by F_a and the assumed directions are indicated by arrows in Fig. 3. For clarity we do not show the normal reaction forces in the figure. Since smooth disk-support interactions are assumed, it follows immediately that $F_8=0$ and the rotational equilibrium condition for g is satisfied identically. Thus, there remain 17 normal and 11 frictional reactions to be determined from 23 equilibrium conditions—again 5 degrees of freedom. As we discussed in the smooth disk case, Sec. II, the degrees

TABLE I. Values for the contact force magnitudes in the four cases, see text. A applies to arrangements shown in Figs. 1 and 2(a). B applies to arrangements shown in Figs. 1, 2(b), and 2(c). C applies to arrangements shown in Fig. 2(e). D applies to arrangements shown in Figs. 1 and 2(f). The entries have been scaled. For the sections labeled with an asterisk, the scale factor is $w/\sqrt{3}$ and for the section with a dagger, the factor is w. No entry indicates the absence of a contact.

	A	B	C	D
R_0	1*	1		1
R_1	0	1		0
R_2	2	3	1	2
R_3	1	0	1	1
R_4	1	0		1
R_5	2	3		2
R_6	0	0	1	1
R_7	0	0	0	0
R_8	3	4	3	3
R_9	1	1		0
R_{10}	2	1		1
R_{11}	2	1	3	3
R_{12}	1	1	1	1
R_{13}	3	4	1	3
$R_{14}-R_L$	$-\frac{3}{2}$	-2		-2
$R_{15}-R_L$	-2	-2		$-\frac{5}{2}$
$R_{16}-R_L$	$-\frac{3}{2}$	-2	$-\frac{1}{2}$	$-\frac{3}{2}$
R_{w1}	$\frac{5}{2}$†	3	$\frac{5}{2}$	3
R_{w2}	$\frac{5}{2}$	2		$\frac{3}{2}$
R_{w3}	$\frac{5}{2}$	2	3	3
R_{w4}	$\frac{5}{2}$	3	$\frac{3}{2}$	$\frac{5}{2}$
R_L	≥ 2*	≥ 2	$=\frac{3}{2}$	$\geq \frac{5}{2}$

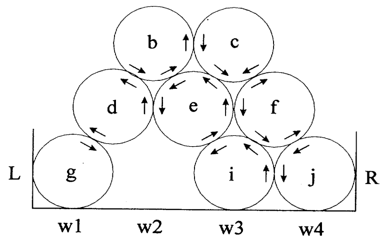

FIG. 3. Pile of eight, rough, rigid disks. The arrows show the directions assumed for the frictional forces. The normal forces have been omitted for clarity. The disk-disk contact labels are shown in Fig. 1.

of freedom are a manifestation of the ways in which one can build the pile. For this pile we shall assume that it has been built in such a fashion that the normal and frictional forces at contact points 1 and 7 vanish, i.e.,

$$R_1=F_1=R_7=F_7=0 . \tag{29}$$

We can then solve for the remaining forces. Using MAPLE, we find

$$R_1=0, \quad F_1=0 , \tag{30}$$

$$R_2=R_3=\frac{w}{2\sqrt{3}}+R_L, \quad F_2=F_3=\tfrac{3}{2}w+\sqrt{3}R_L , \tag{31}$$

$$R_4=R_5=\frac{\sqrt{3}}{2}w+\tfrac{3}{2}R_L, \quad F_4=F_5=-\tfrac{5}{2}w+\tfrac{3}{2}\sqrt{3}R_L , \tag{32}$$

$$R_6=\frac{5}{2\sqrt{3}}w-R_L, \quad F_6=\tfrac{3}{2}w-\sqrt{3}R_L , \tag{33}$$

$$R_7=0, \quad F_7=0 , \tag{34}$$

$$R_8=2R_L, \quad F_8=0 , \tag{35}$$

$$R_{11}=\tfrac{5}{2}\sqrt{3}w-\tfrac{5}{2}R_L, \quad F_{11}=-\tfrac{1}{2}w+\tfrac{1}{2}\sqrt{3}R_L , \tag{36}$$

$$R_{12}=\frac{10}{\sqrt{3}}w-\frac{11}{2}R_L, \quad F_{12}=-w+\tfrac{1}{2}\sqrt{3}R_L , \tag{37}$$

$$R_{13}=-\frac{13}{2\sqrt{3}}w+5R_L, \quad F_{13}=-\tfrac{3}{2}w+\sqrt{3}R_L , \tag{38}$$

$$R_{16}=\frac{w}{\sqrt{3}}, \quad F_{16}=-\tfrac{3}{2}w+\sqrt{3}R_L , \tag{39}$$

$$R_{w1}=w+\sqrt{3}R_L , \tag{40}$$

$$R_{w3}=\tfrac{17}{2}w-3\sqrt{3}R_L , \tag{41}$$

$$R_{w4}=-\tfrac{3}{2}w+2\sqrt{3}R_L , \tag{42}$$

$$R_R=R_L . \tag{43}$$

In contrast to the smooth disk results, Eqs. (5)–(21), the supporting force R_L appears explicitly in the expressions for most of the other R_a.

A simple calculation shows that none of the normal reactions are negative for

$$(13/10)\sqrt{3}w \le R_L \le (20/11)\sqrt{3}w , \tag{44}$$

and at least one normal reaction is negative outside this range. Therefore, the system has physical equilibrium states only if the supporting force R_L satisfies (44). We note two further particular properties. At the lower bound in (44), $R_{13}=0$ and $F_{13}\neq 0$, and at the upper bound, $R_{12}=0$ and $F_{12}\neq 0$. Second, the frictional forces $F_2=F_3=F_{13}=F_{16}=0$ when $R_L=(\sqrt{3}/2)w$, satisfying (44), and $F_4=F_5=0$ when $R_L=5w/(3\sqrt{3})$, again satisfying (44). Thus, we have the following picture. As the supporting force R_L sweeps through the range (44), the normal reactions and frictional forces adjust to satisfy the equilibrium conditions. In particular some of the frictional forces change direction during this process. We emphasize that this adjustment to changes in the supporting force extends throughout the pile to the topmost disks [Eqs. (31) and (32)]. It is clear that, in general, there is a feedback mechanism operating in the equilibrium conditions. The equilibrium forces on the lower disks are affected by the presence of the upper disks and conversely the supporting forces on the upper disks depend on the conditions extant in the lower rows.

The so called top down algorithm has been used to calculate the force distribution in piles of rough, rigid disks, [5–8]. In this method the equilibrium conditions are solved sequentially from the top row down. That is to say the equilibrium condition for each disk is solved using only the previously determined forces on the disks above. Because the conditions in the disks below are ignored, the problem is indeterminate. The impasse is broken by (a) restricting the discussion to the case of at most two point support per disk and (b) assuming limiting friction, i.e., $F_a=\mu_a R_a$, at each contact point. In the pyramid case described above, the ratio F_a/R_a is a function of R_L at each disk-disk contact a. Thus, the static friction coefficient needed at each contact point to make the top-down approach valid varies with the applied force R_L and varies from contact to contact in the pile. Clearly the topdown approach applies to a given set of rough disks only under very special circumstances—particular geometrical arrangements and particular, nonuniform distributions of static coefficients. This means that most physical equilibrium states of the pyramid are not described by the top down approach.

IV. DISCUSSION

To summarize, we have obtained the contact force distributions in a number of different piles of rigid disks in static equilibrium. From these results we find that (a) for a given pile the equilibrium force distribution is not unique but a function of how the pile was constructed, (b) the force distribution depends very sensitively on small geometrical changes in the shape of single disks, and (c) the topdown algorithm can apply only to a very small subset of the permissible equilibrium states for a pile of disks.

The analysis described in earlier sections consists of two steps: (i) finding the algebraic solution to a large number of equilibrium conditions and (ii) determining the physically accessible region in parameter space. With the ready availability and power of modern symbolic mathematics programs such as MAPLE, the first step does not pose a problem for moderate sized piles ("moderate" is determined by the capacity of the computer to hand). Given the algebraic solution from (i), one can then conduct a numerical sweep of parameter space to carry out step (ii).

We conclude with a number of remarks on the possible extension of this work. The physically accessible region in parameter space provides an ensemble of possible states of the pile. By averaging over these states, one can introduce a canonical pile with contact force distribution given by the ensemble averaged contact distribution. We hope to report on canonical piles in a later paper.

It is clear that as the pile increases in size, the algebraic and numerical complexity increases. However, if a thermodynamic limit theorem [14] applies to these hard disk systems, there is a limit to the increase in complexity. If a thermodynamic limit theorem were to hold for a pile of disks, answers to two questions are of particular interest. (a) What properties are independent of size? (b) How large must the system be to reach this limit? In the accompanying paper [15] we describe the properties of an elastic pyramid. We find that the strain fields (normalized to the maximum strains), regarded as a function of position coordinates (normalized to the dimensions of the pile), are independent of size to better than 1 part in 100 for pyramids with 100 or more layers. Based on these results, we might expect in the rigid disk case that the ensemble averaged stress fields (normalized to the maximum stress) regarded as functions of position coordinates (normalized to the dimensions of the pile) would be independent of size in sufficiently large piles. Again extrapolating from the elastic pile case, one might also hope that this thermodynamic limit would be detectable in piles with fewer than 100 layers.

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[1] C. H. Bennett, J. Appl. Phys. **43**, 2727 (1972).
[2] D. J. Adams and A. J. Matheson, J. Chem. Phys. **56**, 1989 (1972).
[3] W. M. Visscher and M. Bolsterli, Nature **239**, 504 (1972).
[4] M. J. Powell, Powder Technol. **25**, 45 (1980).
[5] D. F. Bagster, J. Powder Bulk Solids Technol. **6**, 1 (1982).
[6] G. R. Brooks and D. F. Bagster, J. Powder Bulk Solids Technol. **8**, 18 (1984).
[7] D. F. Bagster and R. Kirk, J. Powder Bulk Solids Technol. **9**, 19 (1985).
[8] E. Li and D. F. Bagster, Powder Technol. **63**, 277 (1990).
[9] K. Liffman, D. Y. C. Chan, and B. D. Hughes, Powder Technol. **72**, 255 (1992).
[10] D. C. Hong, Phys. Rev. E **47**, 760 (1993).
[11] J. Grindlay, Am. J. Phys. **61**, 469 (1993).
[12] J. Smid and J. Novosad, *Particle Technology,* ICH Symp. Series No. 3 (Institute of Chemical Engineers, Rugby, England, 1981).
[13] For the case of a pyramid with N disks along the base, there are $[N(N+1)/2]$ disks, $[N(N+1)-2]$ nontrivial equilibrium conditions, $[N(3N-1)/2]$ contact forces, and hence $[N(N-3)+4]/2$ degrees of freedom.
[14] The thermodynamic limit theorem assures us that certain properties of the system of atoms or particles are independent of the size of the system provided the number of atoms in the system is sufficiently large. The theorem holds for homogeneous systems and special classes of conservative interatomic forces, see, A. Munster, *Statistical Thermodynamics* (Springer, New York, 1969), Vol. 1. Rigid disk systems do not satisfy these conditions: They are not homogeneous and the contact forces are not conservative.
[15] A. H. Opie and J. Grindlay, following paper Phys. Rev. E **51**, 724 (1995).